Lecture 10 on Oct. 10

We already know that with a given curve l connecting 0 and ∞, the log function is well-defined on the space $\mathbb{C} \setminus l$. Now we study its analytic form. From the last lecture, we know that if $e^w = z$, where $w = w_1 + iw_2$, then $w_1 = \log |z|$. Moreover w_2 is one argument of the number $z/|z|$. Supposing that $\mathrm{Arg}(z)$ is the principal argument of z, it holds

$$w_2(z) = \mathrm{Arg}(z) + 2k(z)\pi.$$

Clearly if we assume w_2 is continuous on the set $\mathbb{C} \setminus l$, then $k(z)$ is continous for all $z \in \mathbb{C} \setminus l$. Since $k(z)$ takes its value in $\mathbb{Z}$, therefore we have $k(z) \equiv k_0$ for some integer k_0. Therefore we show that

Proposition 0.1. *on $\mathbb{C} \setminus l$, it holds*

$$\log z = \log |z| + i(\mathrm{Arg}(z) + 2k_0\pi),$$

where k_0 is a constant integer.

By Proposition 0.1 above, we can easily show that

Proposition 0.2. *1. $\log z$ is derivable on $\mathbb{C} \setminus l$;*

2. The derivative of $\log z$ equals to $1/z$.

With the definition of $\log z$, we can introduce the definition of the so-called power functions.

Definition 0.3 (power function)**.** *Given α a complex number, we define*

$$z^\alpha = e^{\alpha \log z}.$$

Clearly z^α depends on the branch of the log function.

Moreover the inverse function of trigo functions and elementary transcendental functions can be given. Now we only consider the inverse function of $\cos z$. By definition, we need solve the equation $\cos w = z$. That is

$$\frac{e^{iw} + e^{-iw}}{2} = z.$$

Equivalently we need solve $(e^{iw})^2 - 2ze^{iw} + 1 = 0$. By quadratic formula, we know that

$$e^{iw} = z \pm \sqrt{z^2 - 1}.$$

Clearly the right-hand side of the above equality is defined on $\mathbb{C} \setminus l$, where l is a curve connecting 0 and ∞. Using the above equality, we proceed to show that

$$iw = \log\left(z \pm \sqrt{z^2 - 1}\right).$$

Clearly now log must be defined a set without a curve l_1. l and l_1 could be identical or different. By the above equality, we show that

$$w = -i \log(z \pm \sqrt{z^2 - 1}).$$

Since $z \pm \sqrt{z^2 - 1}$ are reciprocal with respect to each other, therefore,

$$w = \pm i \log(z + \sqrt{z^2 - 1}).$$

One should know that $\pm$ in the front of the right-hand side of the above equality can be omitted by choosing different branch of log function.